Math 10B with Professor Stankova
Worksheet, Discussion #8; Tuesday, 2/19/2019
GSI name: Roy Zhao

1 Induction

1.1 Examples

1. Prove that $2^n > n^2$ for all $n > 4$.

Solution: The base case is $n = 5$ and we get $2^5 = 32 > 5^2 = 25$. Then assume that $2^n > n^2$ for some $n > 4$. Then, we want to show that $2^{n+1} > (n+1)^2$. We have $(n+1)^2 = n^2 + 2n + 1$ and $(2n+1) < n^2$ for $n > 4$ so we get

$$(n+1)^2 = n^2 + 2n + 1 < n^2 + n^2 < 2^n + 2^n = 2^{n+1}.$$

Thus, by MMI, we have shown $2^n > n^2$ for all $n > 4$.

1.2 Problems

2. Prove that $3^n < n!$ for all $n > 6$.

Solution: This is the same thing as $n \geq 7$. We begin with the base case of $n = 7$. On the left side, we have $3^7 = 2187$ and on the right we have $7! = 5040$ so this is true. Then assume that $3^n < n!$ for some $n \geq 7$. We want to show that $3^{n+1} < (n+1)!$. We can start with the left side (or right) to get

$$3^{n+1} = 3 \cdot 3^n < 3 \cdot n! < (n+1) \cdot n! = (n+1)!.$$

Thus, by MMI, we have shown $3^n < n!$ for all $n > 6$.

3. Prove that $a_n = 3^n + 1$ for $n \geq 1$ is the solution to the recurrence relation $a_{n+1} = 4a_n - 3a_{n-1}$ with $a_1 = 4, a_2 = 10$.

Solution: We have to show our initial conditions now of $a_1 = 4$ and $a_2 = 10$. There are two because our recurrence relation is second order. This is true since $3^1 + 1 = 4$

and $3^1 + 1 = 10$. Then, assume that $a_n = 3^n + 1$ for some n and $n-1$, then we have that

$$\begin{aligned} a_{n+1} = 4a_n - 3a_{n-1} &= 4(3^n + 1) - 3(3^{n-1} + 1) \\ &= 4 \cdot 3^n + 4 - 3^n - 3 \\ &= 3 \cdot 3^n + 1 = 3^{n+1} + 1. \end{aligned}$$

Thus, by MMI, the result is shown for all $n \geq 1$.

2 Probability

2.1 Concepts

4. A **probability space** is a pair (Ω, P) where Ω is called the space of outcomes and P is the probability function. An **outcome** is a single thing that could happen in an "experiment". For example, everything that happened at the super bowl this year. An **event** is usually the thing we care about and can often be described in a single sentence. For instance, it could be "Patriots win" or "Rams only score 3 points". The probability function P is a function that takes events (or sentences) to the probability of that event happening.

2.2 Examples

5. Assume that a telephone number is a 7 digit number that does not begin with 0 or 1. If I pick a random telephone number, what is the probability that it begins with a 9 or ends with a 0? What is the probability space?

Solution: The probability space is all telephone numbers and the event is the subset of telephone numbers that begin with 9 or end with 0. We solve this using PIE by saying that the probability is the probability that it begins with 9, plus the probability that it ends with 0, minus the probability that both are true. The probability that it begins with 9 is $\frac{10^6}{8 \cdot 10^6} = \frac{1}{8}$. The probability that it ends with 0 is $\frac{8 \cdot 10^5}{8 \cdot 10^6} = \frac{1}{10}$. The probability of both being true is $\frac{10^5}{8 \cdot 10^6} = \frac{1}{80}$. Thus, the probability is

$$\frac{1}{8} + \frac{1}{10} - \frac{1}{80} = \frac{17}{80}.$$

6. In the 3 classes I'm taking, each has 3 HW assignments that have to be done in order (for a total of 9 HW assignments). If I randomly pick an order to do these 9 HWs, what is the probability that I actually do each of the 3 HWs in order?

Solution: There are 9! ways to pick in what order I do the homework. Then, in order to find out the number of ways there are to do them in order, first choose the order in which to do the homework if I only care about which class's work I am doing. There are $\binom{9}{3} \cdot \binom{6}{3}\binom{3}{3}$ ways to do this and for each of these ways, there is only 1 way to order the homework for the class so I do them in order. Thus, there are $\binom{9}{3,3,3}$ ways to do these HWs in order. Thus, the probability that I do them in order is $\frac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{9!} = \frac{1}{3!3!3!} = \frac{1}{216}$.

2.3 Problems

7. True **FALSE** The probability function P takes outcomes and outputs a probability for that outcome.

Solution: The probability function takes in events, which are subsets of the probability space and a set of outcomes, and outputs a probability for that event.

8. True **FALSE** When calculating the probability of an event $A \subset \Omega$, we can always take $P(A) = |A|/|\Omega|$.

Solution: This is only true if all the outcomes are equally likely. We assume that for these problems but later on we'll see cases when that isn't necessarily true (for example, in a biased coin the probability of heads and tails are no longer 50%.)

9. I roll 4 6-sided die. What is the probability that the sum of the numbers rolled is 7?

Solution: Let the die be distinct. Then there are 6^4 total die rolls. The number of ways to have the sum be 6 is if $x_1 + x_2 + x_3 + x_4 = 7$, but all numbers are greater than 1 so we can subtract 4 to get $x_1 + x_2 + x_3 + x_4 = 3$ and no restrictions. This is a balls and urns problem and this can be done in $\binom{3+4-1}{3} = \frac{6}{3}$ ways. Thus, the probability is $\frac{\binom{6}{3}}{6^4}$.

10. What is the probability that a 5 card poker hand contains at least 1 king?

Solution: We use complementary counting. The answer is 1 minus the probability that the hand contains no kings. This is

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}.$$

11. I am giving out grades to 60 students randomly (A, B, C, D, F). What is the probability that at least half the class got A's?

Solution: Let x_A, x_B, x_C, x_D, x_F be the number of students that got each grade. Then $x_A + x_B + x_C + x_D + x_F = 60$ and the total number of ways to assign grades is $\binom{60+5-1}{60} = \binom{64}{60}$ ways. The number of ways that half the class got A's is $x_A \geq 30$ so we give out 30 A's and then do the problem again with 30 students to get $\binom{30+5-1}{30} = \binom{34}{30}$ ways. So the probability is

$$\frac{\binom{34}{30}}{\binom{64}{60}}.$$

12. Out of the 14 pants that I own, there are 5 of them that are white. Every day for two weeks, I randomly put on a new pair of pants. What is the probability that I won't wear white pants two days in a row?

Solution: This is like the book marking problem. Since I only care about which pants are white, we can consider all the pants that are white the same and the ones not white all the same. Thus, there are a total of $\binom{14}{5}$ ways to choose an ordering of pants. Then to count the number of ways to do this so that I don't wear white pants two days in a row, I first put down the non-white pants and in the 10 spots next to these 9 pants, I choose 5 of them to wear white pants. Thus, there are $\binom{10}{5}$ ways to do this. This gives a probability of

$$\frac{\binom{10}{5}}{\binom{14}{5}}.$$

Note that if you didn't consider the pants distinguishable, you would get the same answer. This is because there are 14! ways to order which order I will wear the pants. Then once we choose an order of white and nonwhite pants, there are 5! ways to rearrange the white pants and 9! ways to rearrange the non-white pants. Thus, the probability is $\frac{5!9!\binom{10}{5}}{14!}$, which is the same.

13. I roll 5 die for Yahtzee. What is the probability that I get a 5 in a row (12345 or 23456)?

Solution: There are going to be 6^5 different possibilities for die rolls. Then to get 5 in a row, I have to get $1-5$ or $2-6$ and for each of these options, there are 5! ways to have the 5 die get those numbers. Thus, the probability is $\frac{2\cdot 5!}{6^5}$.

14. What is the probability that a 5-card poker hand contains two pairs (but not 4 of a kind)?

Solution: The total number of 5 card hands is $\binom{52}{5}$. Now we want to count the number of hands that have two pairs. First, we need to choose the two values that will be paired, and there are $\binom{13}{2}$ ways to do this. Then for each of those values, there are $\binom{4}{2}$ ways to choose the pair out of the 4 suits. Finally, there are $\binom{44}{1}$ ways to choose our final card (the card cannot be one of the two values that we chose for the pairs). So, the probability is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}.$$